

Exotic Smoothness and Quantum Gravity II: exotic \mathbb{R}^4 , singularities and cosmology

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Abstract. Since the first work on exotic smoothness in physics, it was folklore to assume a direct influence of exotic smoothness to quantum gravity. In the second paper, we calculate the “smoothness structure” part of the path integral in quantum gravity for the exotic \mathbb{R}^4 as non-compact manifold. We discuss the influence of the “sum over geometries to the “sum over smoothness structure. There are two types of exotic \mathbb{R}^4 : large (no smooth embedded 3-sphere) and small (smooth embedded 3-sphere). A large exotic \mathbb{R}^4 can be produced by using topologically slice but smoothly non-slice knots whereas a small exotic \mathbb{R}^4 is constructed by a 5-dimensional h-cobordism between compact 4-manifolds. The results are applied to the calculation of expectation values, i.e. we discuss the two observables, volume and Wilson loop. Then the appearance of naked singularities is analyzed. By using Mostow rigidity, we obtain a justification of area and volume quantization again. Finally exotic smoothness of the \mathbb{R}^4 produces in all cases (small or large) a cosmological constant.

PACS numbers: 04.60.Gw, 02.40.Ma, 04.60.Rt

Submitted to: *Class. Quantum Grav.*

1. Introduction

Since the first papers about exotic smoothness it was folklore to state an influence of exotic smoothness on the state sum (or path integral) for quantum gravity. In our first paper [Asselmeyer-Maluga(2010)] we calculated the “exotic smoothness” contribution to the path integral for a special class of compact 4-manifolds including the K3 surface. The exotic smoothness structure was constructed by knot surgery. Similar results were obtained by Duston [Duston(2009)] for branched covers. The calculation of the path integral has to formally include the exotic smoothness [Pfeiffer(2004)] to relate it to smooth invariants of 4-manifolds. We demonstrated it in our previous paper (to get the Chern-Simons invariant). Unfortunately the most interesting and physically important case of an exotic \mathbb{R}^4 is also the most complicated one which strongly relies on infinite constructions (Casson handles etc.). The appearance of two classes of exotic \mathbb{R}^4 , large and small, complicates the situation. These two classes have their origin in the two main failures in 4-dimensional differential topology which stays in contrast to the topological theory: the smooth h-cobordism theorem and the large class of non-smoothable, topological 4-manifolds. If there is a smooth embedding of a 3-sphere into the exotic \mathbb{R}^4 then one calls it a *small* exotic \mathbb{R}^4 and if not it is a *large* exotic \mathbb{R}^4 . In this paper we will study the effect of the exotic \mathbb{R}^4 on the functional integral for the Einstein-Hilbert action.

In the next section we present some of the physical assumptions as well the definition of an exotic \mathbb{R}^4 . Then we discuss the existence of a Lorentz metric for an exotic \mathbb{R}^4 . Formally the existence of a Lorentz metric is a purely topological question which can be answered positively. On the other hand, global hyperbolicity and all its consequences depend on the standard smoothness of $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ (as smooth product). Therefore an exotic \mathbb{R}^4 must contain naked singularities. We will further analyze these singularities in section 8 to obtain a pairwise structure by the failure of the Whitney’s trick. The main part of this paper is formed by the sections 4 to 6. It starts with a description of large and small exotic \mathbb{R}^4 . Then we discuss the splitting of the action functional (using the diffeomorphism invariance of the Einstein-Hilbert action) according to these descriptions. Finally we calculate the functional integral at first for a particular exotic \mathbb{R}^4 and then for the whole continuous (radial) family. The discussion of observables, like volume and the Wilson loop, in section 7 completes the picture. Especially we confirm the results of Loop quantum gravity, i.e. the quantization of area and volume. All exotic \mathbb{R}^4 have one common property: the appearance of a cosmological constant.

2. Physical Motivation and model assumptions

Einstein’s insight that gravity is the manifestation of geometry leads to a new view on the structure of spacetime. From the mathematical point of view, spacetime is a smooth 4-manifold endowed with a (smooth) metric as basic variable for general relativity. Later on, the existence question for Lorentz structure and causality problems (see [Hawking and Ellis(1994)]) gave further restrictions on the 4-manifold: causality implies non-compactness, Lorentz structure needs a codimension-1 foliation. Usually, one starts with a globally foliated, non-compact 4-manifold $\Sigma \times \mathbb{R}$ fulfilling all restrictions where Σ is a smooth 3-manifold representing the spatial part. But other non-compact 4-manifolds are also possible, i.e. it is enough to assume a non-compact, smooth 4-manifold endowed with a codimension-1 foliation.

All these restrictions on the representation of spacetime by the manifold concept are clearly motivated by physical questions. Among the properties there is one distinguished element: the smoothness. Usually one assumes a smooth, unique atlas of charts covering the manifold where the smoothness is induced by the unique smooth structure on \mathbb{R} . But as discussed in the introduction, that is not the full story. Even in dimension 4, there are an infinity of possible other smoothness structures (i.e. a smooth atlas) non-diffeomorphic to each other. In the following we will specialize to the \mathbb{R}^4 :

Definition 1 *The smoothness structure of \mathbb{R}^4 is called an **exotic smoothness structure** or **exotic \mathbb{R}^4** if it is non-diffeomorphic to the standard smoothness structure (induced from the smooth product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$).*

The implications for physics seem to be obvious because we rely on the smooth calculus to formulate equations of any field theory. Thus different smoothness structures could represent different physical situations leading to different measurable results. But it should be stressed that *exotic smoothness is not exotic physics!* Exotic smoothness is a mathematical possibility which should be further explored to understand its physical relevance.

3. Lorentz metric and global hyperbolicity

Before we start with the construction of the various exotic \mathbb{R}^4 's (large and small), we will discuss some physical implications which are independent of these constructions. Firstly we consider the existence of a Lorentz metric, i.e. a 4-manifold M (the spacetime) admits a Lorentz metric if (and only if) there is a non-vanishing vector field. In case of a compact 4-manifold M we can use the Poincare-Hopf theorem to state: a compact 4-manifold admits a Lorentz metric if the Euler characteristic vanishes $\chi(M) = 0$. But in a compact 4-manifold there are closed time-like curves (CTC) contradicting the causality or more exactly: the chronology violating set of a compact 4-manifold is non-empty (Proposition 6.4.2 in [Hawking and Ellis(1994)]). Non-compact 4-manifold M admits always a Lorentz metric and a special class of these 4-manifolds have an empty chronology violating set. If \mathcal{S} is an acausal hypersurface in M (i.e., a topological hypersurface of M such that no pair of points of M can be connected by means of a causal curve), then $D^+(\mathcal{S})$ is the future Cauchy development (or domain of dependence) of \mathcal{S} , i.e. the set of all points p of M such that any past-inextendible causal curve through p intersects \mathcal{S} . Similarly $D^-(\mathcal{S})$ is the past Cauchy development of \mathcal{S} . If there are no closed causal curves, then \mathcal{S} is a Cauchy surface if $D^+(\mathcal{S}) \cup \mathcal{S} \cup D^-(\mathcal{S}) = M$. But then M is diffeomorphic to $\mathcal{S} \times \mathbb{R}$ [Bernal and Sánchez(2003)]. The existence of a Cauchy surface implies global hyperbolicity, i.e. a spacetime manifold M without boundary is said to be globally hyperbolic if the following two conditions hold:

- (i) *Absence of naked singularities:* For every pair of points p and q in M , the space of all points that can be both reached from p along a past-oriented causal curve and reached from q along a future-oriented causal curve is compact.
- (ii) *Chronology:* No closed causal curves exist (or "Causality" holds on M).

Usually condition 2 above is replaced by the more technical condition "Strong causality holds on M " but as shown in [Bernal and Saánchez(2007)] instead of "strong

causality”, one can write simply the condition ”causality” (and strong causality will hold under causality plus condition 1 above).

Then all (non-compact) 4-manifolds $\mathcal{S} \times \mathbb{R}$ are the only 4-manifolds which admit a globally hyperbolic Lorentz metric, where the product \times has to be a smooth product not only by physical reasons but also because of the claimed result in [Bernal and Sánchez(2003)]. But more is true [Bernal and Saánchez(2005)],

Theorem 1 *If a spacetime (M, g) is globally hyperbolic, then it is isometric to $(\mathbb{R} \times \mathcal{S}, -f \cdot d\tau^2 + g_\tau)$ with a smooth positive function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a smooth family of Riemannian metrics g_τ on \mathcal{S} varying with τ . Moreover, each $\{t\} \times \mathcal{S}$ is a Cauchy slice.*

Furthermore in [Bernal and Sánchez(2006)] it was shown:

- If a compact spacelike submanifold with boundary of a globally hyperbolic spacetime is acausal then it can be extended to a full Cauchy spacelike hypersurface \mathcal{S} of M , and
- for any Cauchy spacelike hypersurface \mathcal{S} there exists a function as in Th. 1 such that \mathcal{S} is one of the levels $\tau = \text{constant}$.

So, what about exotic 4-manifolds? At first the existence of the Lorentz metric is a purely topological condition which will be fulfilled by all non-compact 4-manifolds independent of the smoothness structure. By considering the global hyperbolicity, the picture changes. An exotic spacetime homeomorphic to $\mathcal{S} \times \mathbb{R}$ is not diffeomorphic to $\mathcal{S} \times \mathbb{R}$. The Cauchy surface \mathcal{S} is a 3-manifold with unique smoothness structure (up to diffeomorphisms), the standard structure. So, the smooth product $\mathcal{S} \times \mathbb{R}$ has also the standard smoothness structure. But the diffeomorphism to $\mathcal{S} \times \mathbb{R}$ is necessary for global hyperbolicity. Therefore an *exotic $\mathcal{S} \times \mathbb{R}$ is never globally hyperbolic but admits a Lorentz metric*. Generally we have an exotic $\mathcal{S} \times \mathbb{R}$ with a Lorentz metric such that the projection $\mathcal{S} \times \mathbb{R} \rightarrow \mathbb{R}$ is a time-function (that is, a continuous function which is strictly increasing on future directed causal curves). But then the exotic $\mathcal{S} \times \mathbb{R}$ has no closed causal curves and must contain naked singularities[‡]. We will later see the source of these singularities.

4. Exotic \mathbb{R}^4

In this section we will give some information about the construction of exotic \mathbb{R}^4 . The existence of a smooth embedding $S^3 \rightarrow R^4$ in the exotic \mathbb{R}^4 splits all exotic \mathbb{R}^4 into two classes, large (no embedding) or small.

4.1. Preliminaries: Slice and non-slice knots

At first we start with some definitions from knot theory. A (smooth) knot K is a smooth embedding $S^1 \rightarrow S^3$. In the following we assume every knot to be smooth. Secondly we exclude wilderness of knots, i.e the knot is equivalent to a polygon in \mathbb{R}^3 or S^3 (tame knot). Furthermore, the n -disk is denoted by D^n with $\partial D^n = S^{n-1}$.

Definition 2 Smoothly Slice Knot: *A knot in $\partial D^4 = S^3$ is smoothly slice if there exists a two-disk D^2 smoothly embedded in D^4 such that the image of $\partial D^2 = S^1$ is K .*

[‡] Any non-compact manifold M admits stably causal metrics (that is, those with a time function). So, if M is not diffeomorphic to some product $\mathcal{S} \times \mathbb{R}$, all these (causally well behaved) metrics must contain naked singularities. We thank M. Sánchez for the explanation of this result.

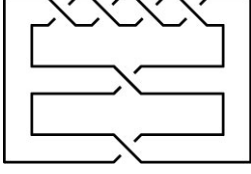


Figure 1. Stevedore's knot 6_1

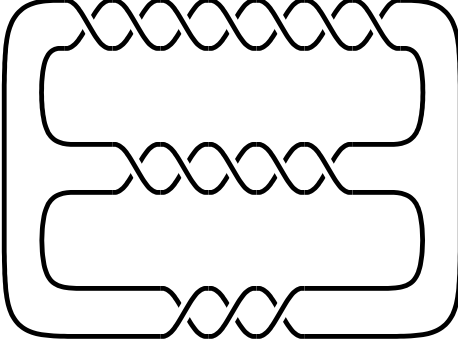


Figure 2. pretzel knot $(-3, 5, 7)$

An example of a slice knot is the so-called Stevedore's Knot (in Rolfsen notation 6_1 , see Fig. 1).

Definition 3 Flat Topological Embedding: Let X be a topological manifold of dimension n and Y a topological manifold of dimension m where $n < m$. A topological embedding $\rho : X \rightarrow Y$ is flat if it extends to a topological embedding $\rho : X \times D^{m-n} \rightarrow Y$.

Topologically Slice Knot: A knot K in ∂D^4 is topologically slice if there exists a two-disk D^2 flatly topologically embedded in D^4 such that the image of ∂D^2 is K .

Here we remark that the flatness condition is essential. Any knot $K \subset S^3$ is the boundary of a disc D^2 embedded in D^4 , which can be seen by taking the cone over the knot. But the vertex of the cone is a non-flat point (the knot is crashed to a point). The difference between the smooth and the flat topological embedding is the key for the following discussion. This innocent looking difference seem to imply that both definitions are equivalent. But deep results from 4-manifold topology gave a negative answer: there are topologically slice knots which are not smoothly slice. An example is the pretzel knot $(-3, 5, 7)$ (see Fig. 2). In [Freedman(1982a)], Freedman gave a topological criteria for topological sliceness: the Alexander polynomial $\Delta_K(t)$ (the best known knot invariant, see [Rolfsen(1976)]) of the knot K has to be one, $\Delta_K(t) = 1$. An example how to measure the smooth sliceness is given by the smooth 4-genus $g_4(K)$ of the knot K , i.e. the minimal genus of a surface F smoothly embedded in D^4 with boundary $\partial F = K$ the knot. This surface F is called the *Seifert surface*. Therefore, if the smooth 4-genus vanishes $g_4(K) = 0$ then the knot K bounds a 2-disk D^2 (surface of genus 0) given by the smooth embedding $D^2 \rightarrow D^4$ so that the image of $\partial D^2 \rightarrow \partial D^4$ is the knot K .

4.2. Large exotic \mathbb{R}^4 and non-slice knots

Large exotic \mathbb{R}^4 can be constructed by using the failure to arbitrarily split of a compact, simple-connected 4-manifold. For every topological 4-manifold one knows how to split this manifold *topologically* into simpler pieces using the work of Freedman [Freedman(1982b)]. But as shown by Donaldson [Donaldson(1983)], some of these 4-manifolds do not exist as smooth 4-manifolds. This contradiction between the continuous and the smooth case produces the first examples of exotic \mathbb{R}^4 [Gompf(1983)]. Unfortunately, the construction method is rather indirect and therefore useless for the calculation of the path integral contribution of the exotic \mathbb{R}^4 . But as pointed out by Gompf (see [Gompf(1985)] or [Gompf and Stipsicz(1999)] Exercise 9.4.23 on p. 377ff and its solution on p. 522ff), large exotic \mathbb{R}^4 can be also constructed by using smoothly non-slice but topologically slice knots. Especially one obtains an explicit construction which will be used in the calculations later.

Let K be a knot in ∂D^4 and X_K the two-handlebody obtained by attaching a two-handle to D^4 along K with framing 0. That means: one has a two-handle $D^2 \times D^2$ which is glued to the 0-handle D^4 along its boundary using a map $f : \partial D^2 \times D^2 \rightarrow \partial D^4$ so that $f(\cdot, x) = K \times x \subset S^3 = \partial D^4$ for all $x \in D^2$ (or the image $im(f) = K \times D^2$ is the solid knotted torus). Let $\rho : X_K \rightarrow \mathbb{R}^4$ be a flat topological embedding (K is topologically slice). For K a smoothly non-slice knot, the open 4-manifold

$$R^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \quad (1)$$

where $\text{int}\rho(X_K)$ is the interior of $\rho(X_K)$, is homeomorphic but non-diffeomorphic to \mathbb{R}^4 with the standard smoothness structure (both pieces are glued along the common boundary ∂X_K). The proof of this fact (R^4 is exotic) is given by contradiction, i.e. let us assume R^4 is diffeomorphic to \mathbb{R}^4 . Thus, there exists a diffeomorphism $R^4 \rightarrow \mathbb{R}^4$. The restriction of this diffeomorphism to X_K is a smooth embedding $X_K \hookrightarrow \mathbb{R}^4$. However, such a smooth embedding exists if and only if K is smoothly slice (see [Gompf and Stipsicz(1999)]). But, by hypothesis, K is not smoothly slice. Thus by contradiction, there exists a no diffeomorphism $R^4 \rightarrow \mathbb{R}^4$ and R^4 is exotic, homeomorphic but not diffeomorphic to \mathbb{R}^4 . Finally, we have to prove that R^4 is large. X_K , by construction, is compact and a smooth submanifold of R^4 . By hypothesis, K is not smoothly slice and therefore X_K can not smoothly embed in \mathbb{R}^4 . By restriction, $D^4 \subset X_K$ and also $\partial D^4 = S^3$ can not smoothly embed and therefore R^4 is a large exotic \mathbb{R}^4 .

4.3. Small exotic \mathbb{R}^4 and Casson handles

Small exotic \mathbb{R}^4 's are again the result of anomalous smoothness in 4-dimensional topology but of a different kind than for large exotic \mathbb{R}^4 's. In 4-manifold topology [Freedman(1982b)], a homotopy-equivalence between two compact, closed, simply-connected 4-manifolds implies a homeomorphism between them (a so-called h cobordism). But Donaldson [Donaldson(1987)] provided the first smooth counterexample, i.e. both manifolds are generally not diffeomorphic to each other. The failure can be localized in some contractible submanifold (Akbulut cork) so that an open neighborhood of this submanifold is a small exotic \mathbb{R}^4 . The whole procedure implies that this exotic \mathbb{R}^4 can be embedded in the 4-sphere S^4 .

The idea of the construction is simply given by the fact that every such smooth h-cobordism between non-diffeomorphic 4-manifolds can be written as a product cobordism except for a compact contractible sub-h-cobordism V , the Akbulut cork.

An open subset $U \subset V$ homeomorphic to $[0, 1] \times \mathbb{R}^4$ is the corresponding sub-h-cobordism between two exotic \mathbb{R}^4 's. These exotic \mathbb{R}^4 's are called ribbon \mathbb{R}^4 's. They have the important property of being diffeomorphic to open subsets of the standard \mathbb{R}^4 . To be more precise, consider a pair (X_+, X_-) of homeomorphic, smooth, closed, simply-connected 4-manifolds.

Theorem 2 *Let W be a smooth h-cobordism between closed, simply connected 4-manifolds X_- and X_+ . Then there is an open subset $U \subset W$ homeomorphic to $[0, 1] \times \mathbb{R}^4$ with a compact subset $C \subset U$ such that the pair $(W \setminus C, U \setminus C)$ is diffeomorphic to a product $[0, 1] \times (X_- \setminus C, U \cap X_- \setminus C)$. The subsets $R_{\pm} = U \cap X_{\pm}$ (homeomorphic to \mathbb{R}^4) are diffeomorphic to open subsets of \mathbb{R}^4 . If X_- and X_+ are not diffeomorphic, then there is no smooth 4-ball in R_{\pm} containing the compact set $Y_{\pm} = C \cap R_{\pm}$, so both R_{\pm} are exotic \mathbb{R}^4 's.*

Thus, remove a certain contractible, smooth, compact 4-manifold $Y_- \subset X_-$ (called an Akbulut cork) from X_- , and re-glue it by an involution of ∂Y_- , i.e. a diffeomorphism $\tau : \partial Y_- \rightarrow \partial Y_-$ with $\tau \circ \tau = Id$ and $\tau(p) \neq \pm p$ for all $p \in \partial Y_-$. This argument was modified above so that it works for a contractible *open* subset $R_- \subset X_-$ with similar properties, such that R_- will be an exotic \mathbb{R}^4 if X_+ is not diffeomorphic to X_- . Furthermore R_- lies in a compact set, i.e. a 4-sphere or R_- is a small exotic \mathbb{R}^4 . In the next subsection we will see how this results in the construction of handlebodies of exotic \mathbb{R}^4 . In [DeMichelis and Freedman(1992)] Freedman and DeMichelis constructed also a continuous family of small exotic \mathbb{R}^4 .

Now we are ready to discuss the decomposition of a small exotic \mathbb{R}^4 by Bizaca and Gompf [Bizaca and Gompf(1996)] by using special pieces, the handles forming a handle body. Every 4-manifold can be decomposed (seen as handle body) using standard pieces such as $D^k \times D^{4-k}$, the so-called k -handle attached along $\partial D^k \times D^{4-k}$ to the boundary $S^3 = \partial D^4$ of a 0-handle $D^0 \times D^4 = D^4$. The construction of the handle body can be divided into two parts. The first part is known as the Akbulut cork, a contractable 4-manifold with boundary a homology 3-sphere (a 3-manifold with the same homology as the 3-sphere). The Akbulut cork A_{cork} is given by a linking between a 1-handle and a 2-handle of framing 0. The second part is the Casson handle CH which will be considered now.

Let us start with the basic construction of the Casson handle CH . Let M be a smooth, compact, simple-connected 4-manifold and $f : D^2 \rightarrow M$ a (codimension-2) mapping. By using diffeomorphisms of D^2 and M , one can deform the mapping f to get an immersion (i.e. injective differential) generically with only double points (i.e. $\#|f^{-1}(f(x))| = 2$) as singularities [Golubitsky and Guillemin(1973)]. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle $D^2 \times D^2$ induced by $f \times id : D^2 \times D^2 \rightarrow M$ from f . Then every double point (or self-intersection) of $f(D^2)$ leads to self-plumbings of the 2-handle $D^2 \times D^2$. A self-plumbing is an identification of $D_0^2 \times D^2$ with $D_1^2 \times D^2$ where $D_0^2, D_1^2 \subset D^2$ are disjoint sub-disks of the first factor disk§. Consider the pair $(D^2 \times D^2, \partial D^2 \times D^2)$ and produce finitely many self-plumbings away from the attaching region $\partial D^2 \times D^2$ to get a kinky handle $(k, \partial^- k)$ where $\partial^- k$ denotes the attaching region of the kinky handle. A kinky handle $(k, \partial^- k)$ is a one-stage tower $(T_1, \partial^- T_1)$ and an $(n+1)$ -stage tower $(T_{n+1}, \partial^- T_{n+1})$ is an n -stage tower union kinky handles $\bigcup_{\ell=1}^n (T_{\ell}, \partial^- T_{\ell})$ where

§ In complex coordinates the plumbing may be written as $(z, w) \mapsto (w, z)$ or $(z, w) \mapsto (\bar{w}, \bar{z})$ creating either a positive or negative (respectively) double point on the disk $D^2 \times 0$ (the core).

two towers are attached along ∂^-T_ℓ . Let T_n^- be $(\text{interior}T_n) \cup \partial^-T_n$ and the Casson handle

$$CH = \bigcup_{\ell=0} T_\ell^-$$

is the union of towers (with direct limit topology induced from the inclusions $T_n \hookrightarrow T_{n+1}$).

The main idea of the construction above is very simple: an immersed disk (disk with self-intersections) can be deformed into an embedded disk (disk without self-intersections) by sliding one part of the disk along another (embedded) disk to kill the self-intersections. Unfortunately the other disk can be immersed only. But the immersion can be deformed to an embedding by a disk again etc. In the limit of this process one "shifts the self-intersections into infinity" and obtains^{||} the standard open 2-handle $(D^2 \times \mathbb{R}^2, \partial D^2 \times \mathbb{R}^2)$.

A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point $*$, having all edge paths infinitely extendable away from $*$. Each edge should be given a label $+$ or $-$. Here is the construction: $\text{tree} \rightarrow CH$. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. Each building block of a Casson handle, the "kinky" handle with n kinks[¶], is diffeomorphic to the n -times boundary-connected sum $\natural_n(S^1 \times D^3)$ (see appendix Appendix A) with two attaching regions. Technically speaking, one region is a tubular neighborhood of band sums of Whitehead links connected with the previous block. The other region is a disjoint union of the standard open subsets $S^1 \times D^2$ in $\#_n S^1 \times S^2 = \partial(\natural_n S^1 \times D^3)$ (this is connected with the next block).

5. The action functional

In this section we will discuss the Einstein-Hilbert action functional

$$S_{EH}(M) = \int_M R \sqrt{g} d^4x \quad (2)$$

of the 4-manifold M and fix the Ricci-flat metric g as solution of the vacuum field equations of the exotic 4-manifold. The main part of our argumentation is additional contribution to the action functional coming from exotic smoothness.

5.1. Large exotic \mathbb{R}^4

In case of the large exotic \mathbb{R}^4 , we consider the decompositions

$$R^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \quad (3)$$

$$\mathbb{R}^4 = (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} \rho(X_K) \quad (4)$$

^{||} In the proof of Freedman [Freedman(1982b)], the main complications come from the lack of control about this process.

[¶] The number of end-connected sums is exactly the number of self intersections of the immersed two handle.

leading to a sum in the action

$$\begin{aligned} S_{EH}(\mathbb{R}^4) &= \int_{\mathbb{R}^4} R\sqrt{g} d^4x = \int_{\mathbb{R}^4 \setminus \text{int}\rho(X_K)} R\sqrt{g} d^4x + \int_{\rho(X_K)} R\sqrt{g} d^4x \\ &= S_{EH}(\mathbb{R}^4 \setminus \text{int}\rho(X_K)) + S_{EH}(\rho(X_K)) \quad . \end{aligned}$$

Because of diffeomorphism invariance of the Einstein-Hilbert action, this decomposition do not depend on the concrete realization with respect to any coordinate system. Therefore we obtain the relation

$$S_{EH}(\mathbb{R}^4 \setminus \text{int}\rho(X_K)) = S_{EH}(\mathbb{R}^4) - S_{EH}(\rho(X_K)) \quad (5)$$

and get a similar relation using (3) between the action $S_{EH}(\mathbb{R}^4)$ of the standard \mathbb{R}^4 and the action $S_{EH}(R^4)$ of the large exotic \mathbb{R}^4

$$\begin{aligned} S_{EH}(R^4) &= S_{EH}(\mathbb{R}^4 \setminus \text{int}\rho(X_K)) + S_{EH}(X_K) \\ &= S_{EH}(\mathbb{R}^4) + S_{EH}(X_K) - S_{EH}(\rho(X_K)) \quad . \end{aligned} \quad (6)$$

The knot is topologically slice (ρ is a flat topological embedding). Therefore the restriction of ρ to the 2-handle $D^2 \times D^2$ in X_K is a topological embedding defining an embedding $\rho' : D^2 \rightarrow D^4$ with $\rho'(\partial D^2) = K$. From the topological point of view, the Seifert surface of K is the disc D^2 with genus 0. Then we obtain using $X_K = D^4 \cup (D^2 \times D^2)$

$$S_{EH}(\rho(X_K)) = S_{EH}(\rho(D^4)) + S_{EH}(\rho(D^2 \times D^2)) = S_{EH}(\rho(D^2 \times D^2))$$

assuming the flatness of the 0-handle D^4 . The product metric (block diagonal metric)

$$ds^2 = g_{D_1} dx_1^2 + g_{D_2} dx_2^2$$

on the image $\rho(D^2 \times D^2) = D_1 \times D_2$ of the 2-handle with $\partial D_1 = K$ induces

$$S_{EH}(\rho(D^2 \times D^2)) = \text{vol}(D_2) \cdot \int_{D_1} R_{D_1} \sqrt{g_{D_1}} d^2x_1 + \text{vol}(D_1) \cdot \int_{D_2} R_{D_2} \sqrt{g_{D_2}} d^2x_2$$

with the curvature scalars R_{D_1}, R_{D_2} . The 2-dimensional integrals

$$\begin{aligned} \int_{D_1} R_{D_1} \sqrt{g_{D_1}} d^2x_1 &= 2\pi \cdot \chi(D_1) = 2\pi \\ \int_{D_2} R_{D_2} \sqrt{g_{D_2}} d^2x_2 &= 2\pi \cdot \chi(D_2) = 2\pi \end{aligned}$$

are by definition the Euler characteristics $\chi(D_1) = 1, \chi(D_2) = 1$ using the topologically sliceness of the knot $K = \partial D_1$. Finally we obtain

$$S_{EH}(\rho(X_K)) = 2\pi \cdot \text{vol}(D_2) + 2\pi \cdot \text{vol}(D_1) \quad .$$

Now we consider the other action $S_{EH}(X_K)$ where we use a non-flat embedding $X_K \hookrightarrow \mathbb{R}^4$. Remember the knot K is smoothly not slice. But then we can only choose the embedding so that the minimal genus $g_4(K)$ of the Seifert surface F is non-zero, i.e. one obtains for the Euler characteristics

$$\int_F R_F \sqrt{g_F} d^2x = 2\pi \cdot (1 - 2g_4(K)) \quad .$$

This genus $g_4(K)$ is an invariant of the knot also known as smooth 4-genus. Importantly the Seifert surface F has negative curvature for $g_4(K) > 0$. A similar argumentation leads to the result

$$S_{EH}(X_K) = 2\pi \cdot \text{vol}(D_2) \cdot (1 - 2g_4(K)) + 2\pi \cdot \text{vol}(F)$$

and finally we have the relation using (6) and the results above

$$S_{EH}(R^4) = S_{EH}(\mathbb{R}^4) - 4\pi \cdot \text{vol}(D_2) \cdot g_4(K) + 2\pi \cdot (\text{vol}(F) - \text{vol}(D_1)) \quad (7)$$

as the correction to the action $S_{EH}(R^4)$ of the large exotic \mathbb{R}^4 . The two surfaces F and D_1 have the same boundary (the knot K) and differ only by the embedding. So, it seems natural to assume the same volume, i.e. $\text{vol}(F) = \text{vol}(D_1)$. Finally we will write this relation in the usual units

$$\frac{1}{\hbar} S_{EH}(R^4) = \frac{1}{\hbar} S_{EH}(\mathbb{R}^4) - \frac{\text{vol}(D_2)}{L_P^2} \cdot 4\pi^2 \cdot g_4(K) \quad (8)$$

This expression looks very simple but the complication is located at the 4-genus $g_4(K)$. Currently there is no simple expression for the calculation. All results show only the existence $g_4(K) \neq 0$ but never calculate the value. So, we are not satisfied with the expression above. We would expect that $g_4(K)$ is related to the map ρ which is certainly related to infinite constructions like the Casson handle. If this speculation is correct then one can interpret the expression above as a non-perturbative calculation.

5.2. Small exotic \mathbb{R}^4

As explained above, a small exotic \mathbb{R}^4 can be decomposed into a compact subset A_{cork} (Akbulut cork) and a Casson handle (see [Biřaca and Gompf(1996)]). Especially this exotic \mathbb{R}^4 depends strongly on the Casson handle, i.e. non-diffeomorphic Casson handles lead to non-diffeomorphic \mathbb{R}^4 's. Thus we have to understand the analytical properties of a Casson handle. In [Kato(2004)], the analytical properties of the Casson handle were discussed. The main idea is the usage of the theory of end-periodic manifolds, i.e. an infinite periodic structure generated by W glued along a compact set A_{cork} to get for the interior

$$\mathbb{R}_\theta^4 = \text{int}(A_{cork} \cup_N W \cup_N W \cup_N \dots)$$

the end-periodic manifold. The definition of an end-periodic manifold is very formal (see [Taubes(1987)]) and we omit it here. All Casson handles generated by a balanced tree have the structure of end-periodic manifolds as shown in [Kato(2004)]. By using the theory of Taubes [Taubes(1987)] one can construct a metric on $\dots \cup_N W \cup_N W \cup_N \dots$ by using the metric on W . Then a metric g in \mathbb{R}_θ^4 transforms to a periodic function \hat{g} on the infinite periodic manifold

$$\tilde{Y} = \dots \cup_N W_{-1} \cup_N W_0 \cup_N W_1 \cup_N \dots$$

where W_i is the building block W at the i th place. Then the action of \mathbb{R}_θ^4 can be divided into two parts

$$S_{EH}(\mathbb{R}_\theta^4) = S_{EH}(A_{cork}) + \sum_i S_{EH}(W_i) \quad (9)$$

and we start with the discussion of the compact part A_{cork} . This part A_{cork} is formally given by a so-called plumbing of two spheres A, B with trivial normal bundles having the algebraic intersection number $A \cdot B = 1$ but an extra pair of intersections (with numbers $+1$ and -1). The whole construction can be simplified

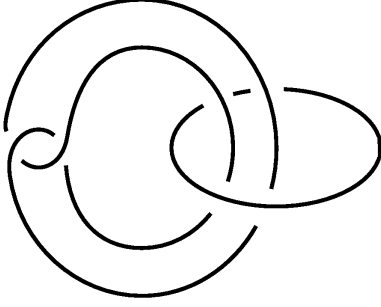


Figure 3. Whitehead link Wh

(see [Gompf and Stipsicz(1999)] p. 361ff) to obtain a diffeomorphism to the Akbulut cork. The boundary of the cork is a homology 3-sphere (a Brieskorn sphere $\Sigma(2, 5, 7)$ see [Akbulut and Kirby(1979)]) with metric of constant curvature. Without loss of generality we choose a homogeneous metric in the interior of the cork $int(A_{cork})$ as well and obtain a constant action

$$S_{EH}(int(A_{cork})) = \lambda_{A_{cork}} \cdot vol(A_{cork}) \quad (10)$$

with respect to the volume of A_{cork} and the curvature $\lambda_{A_{cork}}$. Because of the non-trivial attaching region, the action for W_i has to be non-trivial too. As explained above the simplest part of a Casson handle is the kinky handle given by the n -times boundary-connected sum $\natural_n(S^1 \times D^3)$ where n is the number of self-intersections. Here we will discuss the simplest case $Tree_+$ of a Casson handle with one self-intersection at each level first. It is known by the work of Bizaca [Bizaca(1995)] that this Casson handle admits an exotic smoothness structure. Therefore we assume now a kinky handle with one self-intersection. The attaching region is the disjoint union of $S^1 \times D^2$ which are glued together along $S^1 \times \partial D^2$ to form the boundary $S^1 \times S^2 = \partial(S^1 \times D^3)$. Then the attaching to the boundary of the Akbulut cork ∂A_{cork} is given by a map $\phi : (S^1 \times D^2) \sqcup (S^1 \times D^2) \rightarrow \partial A_{cork}$ where $(S^1 \times D^2) \cup (S^1 \times D^2)$ is mapped to a thickened Whitehead link $N(Wh)$ with

$$\begin{aligned} N(Wh) &= Wh \times D^2 = \phi((S^1 \times D^2) \cup (S^1 \times D^2)) \\ &= \phi(S^1 \times S^2) = \phi(\partial(S^1 \times D^3)) \subset \partial A_{cork} \end{aligned}$$

(see Fig. 3 for Wh). Technically speaking, one attaches two 2-handles along the Whitehead link. Usually one has to define the framing of the link (as degree of ϕ). The value of the action for $S^1 \times D^3$ is unimportant because the main information is contained in the attaching map ϕ and the action $S_{EH}(W_1)$ (where W_1 is attached to A_{cork}) is given by

$$S_{EH}(W_1) = \int_{\phi(\partial(S^1 \times D^3)) \times (0, \epsilon)} R \sqrt{g} d^4 x$$

with an epsilon neighborhood of the attaching map $\phi(\partial(S^1 \times D^3)) \times (0, \epsilon)$. This neighborhood is necessary to represent the framing of the Whitehead link. In our case this framing is zero and we obtain an epsilon neighborhood as product. The introduction of a product metric for this neighborhood

$$ds^2 = d\theta^2 + h_{ik} dx^i dx^k$$

with coordinate θ on $(0, \epsilon)$ and metric h_{ik} on the attaching region $\phi(\partial(S^1 \times D^3)) \subset \partial A_{cork}$. We are using the ADM formalism with the lapse N and shift function N^i to get a relation between the 4-dimensional R and the 3-dimensional scalar curvature $R_{(3)}$ (see [Misner et al.(1973) Misner, Thorne, and Wheeler] (21.86) p. 520)

$$\sqrt{g} R d^4x = N\sqrt{h} (R_{(3)} + \|n\|^2((tr\mathbf{K})^2 - tr\mathbf{K}^2)) d\theta d^3x$$

with the normal vector n and the extrinsic curvature \mathbf{K} . We can arrange that the extrinsic curvature has a fixed value $\mathbf{K} = const$. Then we obtain

$$\int_{\phi(\partial(S^1 \times D^3)) \times (0, \epsilon)} R\sqrt{g} d^4x = \epsilon \int_{\phi(\partial(S^1 \times D^3))} R_{(3)} N\sqrt{h} d^3x$$

with the integral $\epsilon = \int d\theta$. As mentioned above, because of the framing the epsilon do not vanish and we choose a minimal length $\epsilon = L_P$. Finally we have

$$S_{EH}(W_1) = L_P \int_{\phi(\partial(S^1 \times D^3))} R_{(3)} N\sqrt{h} d^3x = L_P \int_{N(Wh)} R_{(3)} N\sqrt{h} d^3x$$

and we integrate over the thickened Whitehead link $N(Wh) \subset \partial A_{cork}$. Now we use the standard trick

$$\partial A_{cork} = (\partial A_{cork} \setminus N(Wh)) \cup N(Wh)$$

to express $N(Wh)$ by ∂A_{cork} and the link complement $\partial A_{cork} \setminus N(Wh)$. Then we obtain for the action

$$S_{EH}(W_1) = L_P \left(\int_{\partial A_{cork}} R_{(3)} N\sqrt{h} d^3x - \int_{\partial A_{cork} \setminus N(Wh)} R_{(3)} N\sqrt{h} d^3x \right)$$

and we have to deal with 3-dimensional Einstein-Hilbert action over ∂A_{cork} and the link complement $\partial A_{cork} \setminus N(Wh)$ only. But as shown by Witten [Witten(8889), Witten(1989), Witten(1991)] this integral

$$\int_{\Sigma} R_{(3)} N\sqrt{h} d^3x = L_{\Sigma} \cdot CS(\Sigma)$$

over the 3-manifold Σ is the Chern-Simons invariant of Σ . The length is set to $L_{\Sigma} = \sqrt[3]{vol(\Sigma)}$. Thus the action is given by

$$S_{EH}(W_1) = L_P \left(\sqrt[3]{vol(\partial A_{cork})} \cdot CS(\partial A_{cork}) - \sqrt[3]{vol(\partial A_{cork} \setminus N(Wh))} \cdot CS(\partial A_{cork} \setminus N(Wh)) \right) \quad (11)$$

for the first level. Beginning with the next level, we have the attachment of the kinky handle to the boundary of the 0-handle, i.e. to the 3-sphere. Therefore we must exchange $\partial A_{cork} K$ by S^3 to obtain for the n th level

$$S_{EH}(W_n) = -L_P \sqrt[3]{vol(S^3 \setminus N(Wh))} \cdot CS(S^3 \setminus N(Wh)) \quad (12)$$

using the vanishing of $CS(S^3) = 0$. Up to now we do not discuss the size of W_n relative to W_{n-1} . As shown by Freedman [Freedman(1982b)], there is a complex network of re-embedding theorems so the 7-stage tower embed into a 6-stage tower etc. Therefore we define the relation

$$S_{EH}(W_n) = \sigma \cdot S_{EH}(W_{n-1}) \quad (13)$$

with the free (regularization) parameter $0 < \sigma < 1$ to reflect the inclusion of higher stage towers into lower stage towers. Finally we obtain for the action

$$\begin{aligned} S_{EH}(\mathbb{R}_\theta^4) &= \lambda_{A_{cork}} \cdot \text{vol}(A_{cork}) + L_P \sqrt[3]{\text{vol}(\partial A_{cork})} \cdot CS(\partial A_{cork}) \\ &\quad - L_P \sqrt[3]{\text{vol}(\partial A_{cork} \setminus N(Wh))} \cdot CS(\partial A_{cork} \setminus N(Wh)) \\ &\quad - L_P \sqrt[3]{\text{vol}(S^3 \setminus N(Wh))} \cdot CS(S^3 \setminus N(Wh)) \cdot \frac{\sigma}{1 - \sigma} \end{aligned}$$

where we used the equations (9,10,11,12,13) and the series

$$\frac{\sigma}{1 - \sigma} = \sum_{i=1}^{\infty} \sigma^i$$

It is obvious that the complexity of the Casson handle is encoded in the series of σ determined by the relation (13). Here we assume a Casson handle with no branching. In the general case we have to include the branching at every level. Then one can encode the tree structure into a general polynomial

$$\sum_{n=1}^{\infty} a_n \sigma^n$$

where the coefficients a_n encode the branching information, i.e. a_n is the number of branchings at the level n . Then we have to choose σ so that the sum converges. We will later come back to this point.

Before we will discuss the path integral, some words about the values of the Chern-Simons invariants.. At first the Akbulut cork K has the boundary $\partial K = \Sigma(2, 5, 7)$, a Brieskorn sphere. The (minimal) Chern-Simons invariant of this homology sphere was calculated in [Fintushel and Stern(1990), Kirk and Klassen(1990), Freed and Gompf(1991)] to be

$$CS(\Sigma(2, 5, 7)) = \frac{9}{280} \bmod 1$$

Usually there is more than one value but only the minimal one corresponds to the Levi-Civita connection (needed in our Einstein-Hilbert action). Using SnapPea of J. Weeks one can also calculate the volume and the Chern-Simons invariant of the complement $S^3 \setminus N(Wh)$. This complement is a hyperbolic 3-manifold. By Mostow rigidity [Mostow(1968)] the volume and the Chern-Simons invariant are topological invariants. Then one obtains for the complement of the Whitehead link

$$\begin{aligned} \text{vol}(S^3 \setminus N(Wh)) &= 3.66386... \\ CS(S^3 \setminus N(Wh)) &= +2.46742... \bmod \pi^2 \end{aligned}$$

For the complement $\partial A_{cork} \setminus N(Wh)$ (obviously it is a hyperbolic 3-manifold) we use a sum decomposition

$$\partial A_{cork} \setminus N(Wh) = \partial A_{cork} \# S^3 \setminus N(Wh)$$

and get for the Chern-Simons invariant

$$\begin{aligned} CS(\partial A_{cork} \# S^3 \setminus N(Wh)) &= CS(\partial A_{cork}) + CS(S^3 \setminus N(Wh)) \\ &= CS(\Sigma(2, 5, 7)) + CS(S^3 \setminus N(Wh)) \end{aligned}$$

Mostow rigidity means also that we have to introduce a fixed length scale L_W for W_1, W_2 which will be scaled for W_n by σ . Then in the usual units we obtain the

action

$$\begin{aligned}
 \frac{1}{\hbar} S_{EH}(\mathbb{R}^4_\theta) &= \frac{1}{\hbar} S_{EH}(\mathbb{R}^4) + \frac{\lambda_{A_{cork}}}{L_P^2} \cdot \text{vol}(A_{cork}) + \frac{\sqrt[3]{\text{vol}(\partial A_{cork})}}{L_P} \cdot CS(\partial A_{cork}) \\
 &\quad - \frac{L_W \cdot \sqrt[3]{\text{vol}(\partial A_{cork} \setminus N(Wh))}}{L_P^2} \cdot (CS(\partial A_{cork}) + CS(S^3 \setminus N(Wh))) \\
 &\quad - \frac{L_W \cdot \sqrt[3]{\text{vol}(S^3 \setminus N(Wh))}}{L_P^2} \cdot CS(S^3 \setminus N(Wh)) \cdot \frac{\sigma}{1-\sigma}
 \end{aligned} \tag{14}$$

for the small exotic \mathbb{R}^4_θ .

6. The functional integral

Now we will discuss the (formal) path integral

$$Z = \int Dg \exp\left(\frac{i}{\hbar} S_{EH}[g]\right) \tag{15}$$

with the action (2) and its conjectured dependence on the choice of the smoothness structure. In the following we will use frames e or connections Γ instead of the metric g . Furthermore we will ignore all problems (ill-definiteness, singularities etc.) of the path integral approach. Then instead of (15) we have

$$Z = \int De \exp\left(\frac{i}{\hbar} S_{EH}[e, M]\right)$$

with the action

$$S_{EH}[e, M] = \int_M \text{tr}(e \wedge e \wedge R)$$

where e is a 1-form (coframe), R is the curvature 2-form R and M is the 4-manifold. Next we have to discuss the measure De of the path integral. Currently there is no rigorous definition of this measure and as usual we assume a product measure. The calculation (8,14) of the action for the two types of exotic \mathbb{R}^4 (large and small) shows an expected independence of the exotic smoothness from a metric. Exotic smoothness of the large/small \mathbb{R}^4 depends only on a continuous parameter (say t) and the path integral (as "sum over geometries and differential structure") should integrate over this parameter as well.

So the action $S_{EH}(R)$ of an exotic \mathbb{R}^4 (denoted by R^4) has the principal structure

$$S_{EH}[e, R^4] = S_{EH}[e, \mathbb{R}^4] + S_{exotic}[t]$$

and we obtain

$$\begin{aligned}
 Z &= \int De \exp\left(\frac{i}{\hbar} S_{EH}[e, R^4]\right) \\
 &= \left(\int De \exp\left(\frac{i}{\hbar} S_{EH}[e, \mathbb{R}^4]\right) \right) \cdot \int Dt \exp\left(\frac{i}{\hbar} S_{exotic}[t]\right)
 \end{aligned}$$

for the path integral. The first part

$$Z_0 = \int_{\text{Geometries}} De \exp\left(\frac{i}{\hbar} S_{EH}[e, \mathbb{R}^4]\right) \tag{16}$$

is the formal integration over the geometries and we are left with the second integration

$$\int Dt \exp\left(\frac{i}{\hbar} S_{exotic}[t]\right) \quad (17)$$

by varying the differential structure of \mathbb{R}^4 . As explained above, there are two possible classes of exotic \mathbb{R}^4 , the large and the small exotic \mathbb{R}^4 . So the parameter has a different meaning. In the large case, the parameter is the radius of the continuous family of large exotic \mathbb{R}^4 eliminating the geometrical dependence in the action (8) (see the discussion below). Because of the Mostow rigidity [Mostow(1968)], the contribution of the small exotic \mathbb{R}^4 to the action (14) is purely topological (except for the volume $vol(K)$ giving only a numerical shift in the action). Finally we have,

Proposition 1 *The (formal) path integral Z of an exotic \mathbb{R}^4 splits into a product of two path integrals*

$$Z = \int De \exp\left(\frac{i}{\hbar} S_{EH}[e, R^4]\right) = Z_0 \cdot \int Dt \exp\left(\frac{i}{\hbar} S_{exotic}[t]\right) \quad (18)$$

i.e. the exotic part is independent of a frame or metric.

6.1. Large exotic \mathbb{R}^4

In subsection 4.2 we discuss the construction of a large exotic \mathbb{R}^4 by using a **fixed** topologically but non-smoothly sliced knot K . Thus by using the relation (8) we obtain

$$Z = Z_0 \cdot \exp\left(-i \frac{vol(D_2)}{L_P^2} \cdot 4\pi^2 \cdot g_4(K)\right) \quad (19)$$

Here we obtain only countable many large exotic \mathbb{R}^4 in this way. To distinguish uncountable many exotic \mathbb{R}^4 's one has to use the following construction. Let R_K^4 be an exotic \mathbb{R}^4 constructed from a topologically but non-smoothly sliced knot K . Now fix a homeomorphism $h : \mathbb{R}^4 \rightarrow R_K^4$ and let $R_r \subset R_K^4$ be the image of the open balls of radius r centered at 0 in \mathbb{R}^4 with $R_\infty = R_K^4$. Each R_t inherits a smooth structure as an open subset of R_K^4 [Quinn(1982)]. By the work of Freedman and Quinn [Freedman and Quinn(1990)] any homeomorphism between smooth 4-manifolds is isotopic to one which is a local diffeomorphism near a preassigned 1-complex (one axis e.g. the non-negative x_1 -axis). Using this result, one can show (see Theorem 9.4.10 in [Gompf and Stipsicz(1999)]) that R_s and R_t are non-diffeomorphic for $0 < s < t < \infty$. The proof based on the fact that there is a compact 4-manifold $K \subset R_t$ which cannot be embedded in R_s . In the construction of subsection 4.2 we used a decomposition (1) with respect to a topologically flat embedding $\rho : X_K \rightarrow \mathbb{R}^4$ of a 2-handle body X_K . This submanifold X_K is compact and the exotic smoothness structure is determined by the smooth failure of this embedding. But then the 4-ball $D^4 \subset X_K$ cannot be embedded. Conversely the image $\rho(X_K)$ is a compact manifold as well and can be surrounded by a 4-ball $\rho(X_K) \subset D^4$ (Heine-Borel theorem). Therefore it is enough to consider a scaling of X_K by the radius t to express the radius family R_t defined above. In the state sum (19) above, one has to replace $vol(D_2)$ by t^2 to express this scaling. For a fixed knot K (topologically slice but smoothly non-slice) and a fixed parameter we have the state sum

$$Z_t = Z_0 \cdot \exp\left(-i \frac{t^2}{L_P^2} \cdot 4\pi^2 \cdot g_4(K)\right)$$

For the full path integral we obtain

$$Z_{large} = Z_0 \cdot \frac{1}{L_P} \int_0^\infty \exp \left(-i \frac{t^2}{L_P^2} \cdot 4\pi^2 \cdot g_4(K) \right) dt$$

in units of the Planck length and finally (using the Fresnel integral)

$$Z_{large} = Z_0 \cdot \frac{e^{-i\pi/4}}{2\pi\sqrt{g_4(K)}} \quad (20)$$

Especially the contribution of the large exotic \mathbb{R}^4 is independent of a scale, i.e. it is (differential-)topological invariant.

6.2. Small exotic \mathbb{R}^4

Now we use our technique to represent the small exotic \mathbb{R}^4 using the decomposition of subsection 4.3. The final result for the action was formula (14). Then we can read the expression for the state sum

$$\begin{aligned} Z = Z_0 \cdot \exp & \left(i \frac{\lambda_{A_{cork}}}{L_P^2} \cdot \text{vol}(A_{cork}) + i \cdot \Lambda_{\partial A_{cork}} \cdot CS(\partial A_{cork}) \right) \cdot \\ & \cdot \exp \left(-i \cdot \Lambda_{\partial A_{cork}, Wh} \cdot (CS(\partial A_{cork}) + CS(S^3 \setminus N(Wh))) \right) \\ & \cdot \exp \left(-i \cdot \Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh)) \cdot \frac{\sigma}{1-\sigma} \right) \end{aligned} \quad (21)$$

with the scaling parameters

$$\begin{aligned} \Lambda_{\partial A_{cork}} &= \frac{\sqrt[3]{\text{vol}(\partial A_{cork})}}{L_P} \\ \Lambda_{\partial A_{cork}, Wh} &= \frac{L_W \cdot \sqrt[3]{\text{vol}(\partial A_{cork} \setminus N(Wh))}}{L_P^2} \\ \Lambda_{S^3, Wh} &= \frac{L_W \cdot \sqrt[3]{\text{vol}(S^3 \setminus N(Wh))}}{L_P^2} \end{aligned} \quad (22)$$

As remarked above, the Whitehead link Wh is an hyperbolic link, i.e. the knot complements $S^3 \setminus N(Wh)$ and $\partial A_{cork} \setminus N(Wh)$ are hyperbolic 3-manifolds (with boundary the disjoint union of two tori). It is not an unexpected result that the contribution of the small exotic \mathbb{R}^4 is a topological invariant (Chern-Simons invariant) by fixed scaling parameters Λ . The variation of the Casson handle produces other small exotic \mathbb{R}^4 . The whole problem was analyzed in [DeMichelis and Freedman(1992)] using the so-called design of a Casson handle, i.e. a singular parametrization of all Casson handles by a binary tree. As shown by Freedman [Freedman(1982b)], the design forms a continuous set (Cantor continuum)⁺. Then to every real number^{*} in $[0, 1]$ there is a Casson handle. In the formula above, the simplest Casson handle is represented by

⁺ This kind of Cantor set is given by the following construction: Start with the unit Interval $S_0 = [0, 1]$ and remove from that set the middle third and set $S_1 = S_0 \setminus (1/3, 2/3)$. Continue in this fashion, where $S_{n+1} = S_n \setminus \{\text{middle thirds of subintervals of } S_n\}$. Then the Cantor set $C.s.$ is defined as $C.s. = \bigcap_n S_n$. With other words, if we using a ternary system (a number system with base 3), then we can write the Cantor set as all sequences containing only 0 or 2 after the decimal point.

^{*} According to [DeMichelis and Freedman(1992)], there is a collection of parameter values (representing the Casson handle) with the cardinality of the continuum in the Zermelo-Fraenkel set theory with choice so that the corresponding small exotic \mathbb{R}^4 's are pairwise non-diffeomorphic.

the expression $\frac{\sigma}{1-\sigma}$ and we have to replace it by the real number $t \in [0, 1]$. Then the last term in (21) is now replaced by

$$\int_0^1 \exp(-i \cdot \Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh)) \cdot t) dt = \frac{e^{i\pi/2} (\exp(-i \Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh))) - 1)}{\Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh))}$$

and therefore we obtain finally

$$\begin{aligned} Z_{small} = & Z_0 \cdot \exp\left(i \frac{\lambda_{A_{cork}}}{L_P^2} \cdot \text{vol}(A_{cork}) + i \cdot \Lambda_{\partial K A_{cork}} \cdot CS(\partial A_{cork})\right) \cdot \\ & \cdot \exp(-i \cdot \Lambda_{\partial A_{cork}, Wh} \cdot (CS(\partial A_{cork}) + CS(S^3 \setminus N(Wh)))) \\ & \cdot \frac{e^{i\pi/2} (\exp(-i \Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh))) - 1)}{\Lambda_{S^3, Wh} \cdot CS(S^3 \setminus N(Wh))} \end{aligned} \quad (23)$$

Then the contribution of the small exotic \mathbb{R}^4 is independent of a scale again, i.e. it is (differential-)topological invariant.

Proposition 2 *Exotic smoothness contributes to the state sum of quantum gravity for all exotic \mathbb{R}^4 .*

7. Observables

Any consideration of quantum gravity is incomplete without considering observables and its expectation values. Here we consider two kinds of observables:

- (i) Volume
- (ii) holonomy along open and closed paths (Wilson loop)

The expectation value for the volume can be calculated only for compact submanifolds of the \mathbb{R}^4 by using the decomposition of the large R_K^4 or small \mathbb{R}_θ^4 exotic \mathbb{R}^4

$$\begin{aligned} R_K^4 &= (\mathbb{R}^4 \setminus \text{int}\rho(X_K)) \cup_{\partial X_K} X_K \\ \mathbb{R}_\theta^4 &= \text{int}(A_{cork} \cup_N W \cup_N W \cup_N \dots) \end{aligned}$$

explained in section 4. Let $D \subset M$ be a submanifold in $M = R_K^4, \mathbb{R}_\theta^4$ with volume $\text{vol}(D)$ (where its meaning depends on the dimension of D). Let

$$\langle \text{Vol}(D) \rangle_0 = \frac{\int De_G \text{Vol}(D, e_G) \exp\left(\frac{i}{\hbar} S_{EH}[e, M]\right)}{\int De_G \exp\left(\frac{i}{\hbar} S_{EH}[e, M]\right)}$$

be the expectation value of the volume w.r.t. the geometry. At first we assume that D is 1-dimensional, i.e. $D = [0, 1]$ or $D = S^1$. In this case, there is no topological restriction and we obtain any value of $\text{vol}(D)$ (remember M is simple connected and so every loop is contractable). The case of a surface is the first non-trivial example. So, we consider a closed surface D of genus g (the surface with boundary can be simply obtained from this case by removing disks). For the large exotic \mathbb{R}^4 denoted by R_K^4 we have to consider two cases:

- (i) D lies at X_K or
- (ii) D lies at $\mathbb{R}^4 \setminus \text{int}\rho(X_K)$

In the first case, the surface D is confined by the handle body X_K . By using the action (8) we can express the expectation value of the volume by a formal variation of $1/L_P^2$ to get

$$\begin{aligned}\langle Vol(D) \rangle_0 &= \frac{\delta \ln Z}{\delta(1/L_P^2)} = \frac{1}{Z} \int De_G \frac{i\partial S_{EH}}{\partial(1/L_P^2)} \exp\left(\frac{i}{\hbar} S_{EH}[e, M]\right) \\ &= vol(D_2) 4\pi^2 g_4(K)\end{aligned}$$

The disk is confined by X_K as expected. Especially the whole effect vanishes for the standard \mathbb{R}^4 with $g_4(K) = 0$. Furthermore for the radial family of large exotic \mathbb{R}^4 , the volume $vol(D_2)$ is related to the radius parameter r by $r^2 = vol(D_2)$. Then the size of $vol(D_2)$ is fixed for one smoothness structure (up to diffeomorphisms), i.e. *we obtain a quantized area of a surface D in units of $vol(D_2) \cdot g_4(K) \cdot 4\pi^2$* . For the second case we do not get any restriction on the volume (or better area) of D .

There is a similar effect for the small exotic \mathbb{R}^4 . Here the situation is more complicate. For instance, the surface D can be inside of the ∂A_{cork} or in one of the periodic pieces W_i . But the three scaling parameters Λ defined by (22) must be quantized. Here one argues via a consistent quantum field theory based on the Chern-Simons action (see Witten [Witten(1991)]) to show the quantization of the parameters. As an example we consider the quantized parameter

$$\Lambda_{S^3, Wh} = \frac{L_W \cdot \sqrt[3]{vol(S^3 \setminus N(Wh))}}{L_P^2} \in \mathbb{N}$$

The volume $vol(S^3 \setminus N(Wh))$ is a topological invariant by Mostow rigidity (the Whitehead links is hyperbolic). Therefore the length scale L_W of the periodic pieces are also quantized. But we remark that for all 1-dimensional submanifolds there is no restriction, i.e. the length of these submanifolds is not quantized. Finally we obtain:

Proposition 3 *The area and the volume of a 3-dimensional submanifold must be quantized in units of $\Lambda \cdot L_P^2$ or $\Lambda \cdot L_P^3$, respectively. The factor Λ depends on the concrete smoothness structure.*

Now we discuss the holonomy

$$hol(\gamma, \Gamma) = Tr \left(\exp \left(i \int_{\gamma} \Gamma \right) \right)$$

along a path γ w.r.t. the connection Γ , as another possible observable. The spacetime M is simple-connected. Thus every path can be deformed to another path. Especially, every closed path is the boundary of a disk (every knot in a 4-space is smoothly deformable (=isotopic) to the unknot). The embedding of this disk is the non-trivial task, i.e. an immersed disk is the best possible alternative to an embedding. But then this disk has self-intersections in the interior which are the source of singularities. Another possibility is the existence of a surface F (the Seifert surface) of non-zero genus with ∂F equal to the closed curve. Usually the exotic \mathbb{R}^4 splits into 4-dimensional pieces W with $\partial W \neq \emptyset$. Therefore we embed (or immerse) the *closed curve* $\gamma = \partial F$ into ∂W and the surface F into W . For the calculation of the expectation value, we have to integrate over the connections in the path integral

$$\langle W(\gamma) \rangle = \frac{\int D\Gamma W(\gamma) \exp\left(\frac{i}{\hbar} S_{EH}(R^4, \Gamma)\right)}{\int D\Gamma \exp\left(\frac{i}{\hbar} S_{EH}(R^4, \Gamma)\right)}$$

with the Wilson loop

$$W(\gamma) = \text{Tr} \left(\exp \left(i \int_{\gamma} \Gamma \right) \right)$$

for the exotic R^4 . For large exotic \mathbb{R}^4 's we do not find any interesting value for the expectation value $\langle W(\gamma) \rangle$ except the usual expression for standard \mathbb{R}^4 . The reason could be that the action of the large exotic \mathbb{R}^4 is only corrected by a constant term in comparison to the action of the standard \mathbb{R}^4 . From the classical point of view, only solutions with a non-trivial curvature will be preferred or the large exotic \mathbb{R}^4 is always curved (in agreement with [Ślaskowski(2001)]). But this effect cannot be localized at some holonomy along closed curve. In contrast small exotic \mathbb{R}^4 have different properties. Now we have two different contributions:

- (i) The closed curve lies at ∂A_{cork} (and the Seifert surface in A_{cork}), or
- (ii) the closed curve lies at ∂W (and the Seifert surface in W).

The first case leads to an integral

$$\langle W(\gamma) \rangle = \frac{\int D\Gamma W(\gamma) \exp(i \cdot (\Lambda_{\partial A_{\text{cork}}} - \Lambda_{\partial A_{\text{cork}}, Wh}) \cdot CS(\partial A_{\text{cork}}, \Gamma))}{\int D\Gamma \exp\left(\frac{i}{\hbar} S_{EH}(R^4, \Gamma)\right)}$$

so that we obtain a knot invariant

$$\langle W(\gamma) \rangle = \text{generalized Jones polynomial } J_{\gamma}(q)$$

of the closed curve γ in ∂A_{cork} where we have

$$q = \exp \left(\frac{2\pi i}{2 + (\Lambda_{\partial A_{\text{cork}}} - \Lambda_{\partial A_{\text{cork}}, Wh})} \right)$$

The second case is very similar and we obtain the principal result

$$\langle W(\gamma) \rangle = \frac{\int D\Gamma W(\gamma) \exp(i \cdot \ell \cdot CS(S^3 \setminus N(Wh), \Gamma))}{\int D\Gamma \exp\left(\frac{i}{\hbar} S_{EH}(R^4, \Gamma)\right)}$$

where the constant ℓ is a combination of the scaling parameters. But the corresponding quantum field theory is only consistent, if the coefficients of the Chern-Simons terms are integer valued. This fact confirms again the Proposition 3.

8. Naked singularities and the failure of the Whitney trick

In this section we will discuss the appearance of naked singularities in exotic \mathbb{R}^4 . The Cauchy surface of the standard \mathbb{R}^4 is given by \mathbb{R}^3 so that $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1$ in agreement with the discussion above. In contrast, any exotic \mathbb{R}^4 cannot be split like $(3\text{-manifold} \times \mathbb{R})$.

To visualize the problem, we consider the following toy model: a non-trivial surface (see Fig. 4) connecting two circles which can be deformed to the usual cylinder. This example can be described by the concept of a cobordism. A cobordism (W, M_1, M_2) between two n -manifolds M_1, M_2 is a $(n+1)$ -manifold W with $\partial W = M_1 \sqcup M_2$ (ignoring the orientation). Then there exists a smooth function $f : W \rightarrow [0, 1]$ such that $f^{-1}(0) = M_1$, $f^{-1}(1) = M_2$. By general position, one can assume that f is a Morse function and such that all critical points occur in the interior of W . In this setting f is called a Morse function on a cobordism. For every critical point of f (vanishing first derivative) one adds a handle $D^k \times D^{n-k}$. In our example

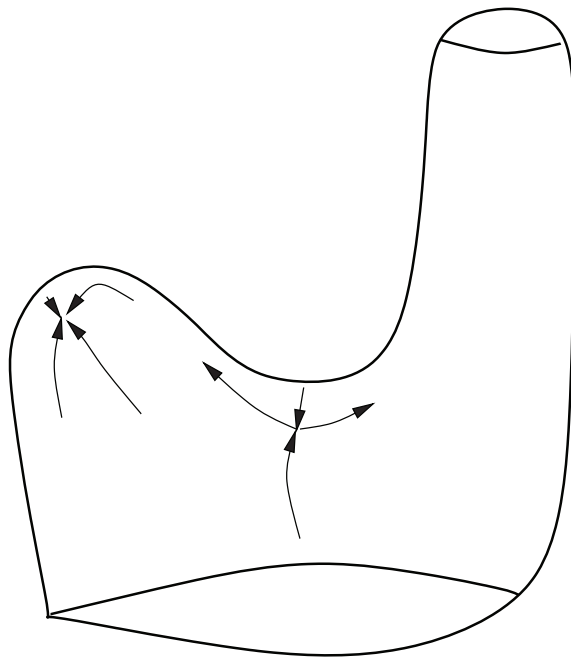


Figure 4. two naked singularities

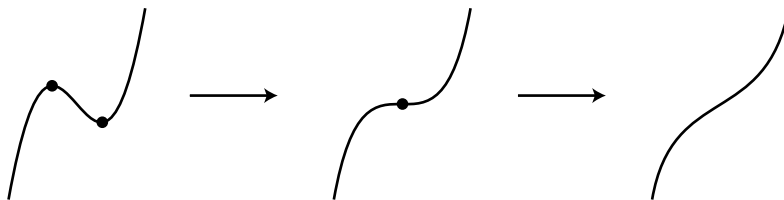


Figure 5. killing a 0- and a 1-handle

in Fig. 4, we add a 2-handle $D^2 \times D^0$ (the maximum) and a 1-handle $D^1 \times D^1$ (the saddle). But obviously this cobordism is diffeomorphic to the trivial one $S^1 \times [0, 1]$ because the two boundary components are diffeomorphic to each other. Therefore the 2-/1-handle pair is "killed" in this case. The 2-handle and the 1-handle differ in one direction where the Morse function has a maximum for the 2-handle and a minimum for the 1-handle. The left graph of Fig. 5 visualizes this fact. Furthermore the sequence of graphs from the left to right presents the process to "kill" the handle pair. In the definition of the cobordism, there is no restriction on the two manifolds M_1, M_2 , i.e. one can consider a cobordism between two non-homeomorphic manifolds. An example is a cobordism between one circle S^1 and the disjoint union of two circles $S^1 \sqcup S^1$ (the pair of pants). But in the discussion above (see section 3) we considered always

a special class of cobordisms, where the two manifolds M_1, M_2 are homeomorphic to each other. Mathematically we have to discuss h-cobordisms between 3-manifolds. Because of the homeomorphism between M_1 and M_2 , the h-cobordism must contain any handle in the interior. Usually the construction of a h-cobordism will produce also handles in the interior. But these handles can be killed where the details of the construction can be found in [Milnor(1965)]. Here we will give only some general remarks. Any $0 - / 1$ -handle pair as well any $n - / (n + 1)$ -handle pair (remember the h-cobordism is $n + 1$ -dimensional) can be killed by a general procedure. The killing of a $k - / (k + 1)$ -handle pair depends on a special procedure, the Whitney trick. For 4- and 5-dimensional h-cobordisms (between 3- and 4-manifolds, respectively) we cannot use the Whitney trick. This failure lies at the heart of the problem to classify 3- and 4-manifolds.

Now we will specialize to a 4-dimensional h-cobordism between 3-manifolds. Then we can kill the $0 - / 1$ - and the $3 - / 4$ -handle pair of the h-cobordism. Then we are left with pairs of 2-handles. If the Whitney trick works in this case, we can kill these pairs of handles. But it is known that the Whitney trick only works topologically. But the existence of exotic $S^3 \times \mathbb{R}$'s (as non-compact examples) gave counterexamples, so that the pairs of 2-handles never cancel each other. The critical point of the Morse function with index 2 (the Morse function has a minimum in two directions (saddle point)) corresponds to the 2-handle. Each pair of 2-handles is connected to each other, i.e. the directions representing the minimum of a 2-handle are connected with the directions representing the maximum of the other 2-handle. Therefore we get

Proposition 4 *The naked singularities of an exotic \mathbb{R}^4 are pairs of 2-handles which cancel topologically but not smoothly by the failure of the Whitney trick.*

9. Cosmological consequences

The global structure of the spacetime \mathbb{R}^4 is greatly influenced by the smoothness structure. Therefore it seems natural to obtain cosmological results from our calculation of the functional integral above. For the large exotic \mathbb{R}^4 we obtained the term (see (8))

$$-\frac{\text{vol}(D_2)}{L_P^2} \cdot 4\pi^2 \cdot g_4(K)$$

as the correction of the action $S_{EH}(\mathbb{R}^4)$. This term can be simply interpreted as the cosmological constant term. At first we remark that the spacetime \mathbb{R}^4 contains the big bang singularity. The removal of this singularity (say at 0) leads to

$$\mathbb{R}^4 \setminus \{0\} = S^3 \times \mathbb{R}$$

i.e. we assume a compact spatial component S^3 of the cosmos. But then we have set for the term above

$$-\frac{\text{vol}(D_2)}{L_P^2} \cdot 4\pi^2 \cdot g_4(K) = \int_{S^3 \times [0,1]} \Lambda_{\text{cosmo}} \sqrt{g} d^4x$$

where we integrate over the time period since the big bang, i.e. the cosmological constant is given by

$$\Lambda_{\text{cosmos}} = -\frac{\text{vol}(D_2)}{L_P^2 \cdot \text{vol}(S^3 \times [0,1])} \cdot 4\pi^2 \cdot g_4(K) < 0$$

a negative number.

The case of a small exotic \mathbb{R}^4 is more interesting. The action (14) contains the term

$$\frac{\lambda_{A_{cork}}}{L_P^2} \cdot \text{vol}(A_{cork})$$

which can be similarly interpreted as cosmological constant term. In contrast to the large exotic \mathbb{R}^4 , we have now a direct model of cosmos. The compact submanifold A_{cork} in the construction of the small exotic \mathbb{R}^4 can serve as a model for the cosmic evolution. This submanifold A_{cork} is contractable with a homology 3-sphere as boundary. Then the cosmological constant is given by

$$\Lambda_{cosmos} = \lambda_{A_{cork}} \sim \frac{1}{\sqrt{\text{vol}(A_{cork})}}$$

Therefore exotic smoothness can be the appropriate view to understand the dark energy.

10. Conclusion

In this paper we discussed the influence of exotic smoothness on the functional integral of the Einstein-Hilbert action. Then we obtain a bunch of results:

- the appearance of naked singularities in exotic \mathbb{R}^4 ,
- any naked singularity is a saddle point (of index 2, i.e. two directions are a minimum) and we have only an even number of it,
- area and volume quantization by using Mostow rigidity [Mostow(1968)] agreeing with results in Loop quantum gravity [Rovelli and Smolin(1995)],
- the appearance of a cosmological constant.

This is only the beginning of a systematic analysis of exotic \mathbb{R}^4 's. Interestingly, there are also rich connections between quantization, non-commutative geometry and exotic smoothness [Asselmeyer-Maluga and Król(2010)].

Finally we can support the physically motivated conjecture that quantum gravity depends on exotic smoothness.

Acknowledgments

This work was partly supported (T.A.) by the LASPACE grant. Many thanks to Carl H. Brans for nearly infinite many discussions about the physics of exotic 4-manifolds. The authors acknowledged for all mathematical discussions with Duane Randall and Terry Lawson. The section about global hyperbolicity is meanly inspired by the discussion with Miguel Sánchez. Many thanks for the enlightened remarks.

Appendix A. Connected and boundary-connected sum of manifolds

Now we will define the connected sum $\#$ and the boundary connected sum \natural of manifolds. Let M, N be two n -manifolds with boundaries $\partial M, \partial N$. The *connected sum* $M \# N$ is the procedure of cutting out a disk D^n from the interior $\text{int}(M) \setminus D^n$ and $\text{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \sqcup \partial M$ and $S^{n-1} \sqcup \partial N$, respectively, and gluing them together along the common boundary component S^{n-1} . The boundary

$\partial(M\#N) = \partial M \sqcup \partial N$ is the disjoint sum of the boundaries $\partial M, \partial N$. The *boundary connected sum* $M\natural N$ is the procedure of cutting out a disk D^{n-1} from the boundary $\partial M \setminus D^{n-1}$ and $\partial N \setminus D^{n-1}$ and gluing them together along S^{n-2} of the boundary. Then the boundary of this sum $M\natural N$ is the connected sum $\partial(M\natural N) = \partial M \# \partial N$ of the boundaries $\partial M, \partial N$.

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